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NONLINEAR STRUCTURAL DYNAMICS EQUATION IN FINITE DISPLACEMENTS FOR THREE-DIMENSIONAL VISCOELASTIC ROTATING STRUCTURES FOR SMALL GEOMETRICAL PERTURBATIONS.

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Abstract. *This paper deals with small geometrical and mechanical perturbations in nonlinear structural dynamics of three-dimensional viscoelastic rotating structures for finite displacements. The objective of the paper is to present the nonlinear equations of the problem written in the rotating frame. The constitutive equation of the material is given for its natural configuration (structure at rest and without prestress). The three-dimensional viscoelasticity theory (stresses depend on actual and past strains) with finite displacements of B.D. Coleman and W. Noll (1961) is used without taking into account the material nonlinearities. The reference configuration is defined as the stationary configuration corresponding to the prestress configuration under gyroscopic forces and stationary parts of the external forces applied to the structure in the rotating frame. Then, for small geometrical and mechanical perturbations (mass density, constitutive equation coefficients) applied to the natural configuration, nonlinear equations around the reference configuration are derived in the time domain and then, the corresponding linearized equations are deduced in the frequency domain and reduced using the Ritz-Galerkin method. This paper shows that the equations obtained for geometrical perturbations are not self-evident.*

Keywords: *Structural dynamics, Rotating structures, Nonlinear elasticity, Geometrical perturbations.*

1. NONLINEAR EQUATIONS IN THE ROTATING FRAME OF A ROTATING STRUCTURE.

Physical space $\mathbb{R}^3$ is referred to a Cartesian reference system $(\mathbf{e}_{0,1}, \mathbf{e}_{0,2}, \mathbf{e}_{0,3})$ with origin O denoted as $\mathcal{R}_0$. The rotating frame (related to the structure in rotation) denoted as $\mathcal{R}_1$ is defined by origin O and a direct orthonormal basis $(\mathbf{e}_{1,1}(t), \mathbf{e}_{1,2}(t), \mathbf{e}_{1,3}(t))$ which is deduced from $(\mathbf{e}_{0,1}, \mathbf{e}_{0,2}, \mathbf{e}_{0,3})$ by a rotation represented by an orthogonal (3×3) real matrix $[Q(t)]$ and is such that

$$\mathbf{e}_{1,p}(t) = [Q(t)]\, \mathbf{e}_{0,p} \quad , \quad p = 1, 2, 3 \quad . \tag{1}$$

In $\mathcal{R}_0$ and at time t, the deformed configuration of the structure occupies a domain denoted as $\widetilde{\Omega}(t)$ with boundary $\widetilde{\Gamma}(t) \cup \widetilde{\Sigma}(t)$. We impose on $\widetilde{\Gamma}(t)$ a rigid-body displacement field defined by the rotation associated with $[Q(t)]$. In order to describe the mouvement of the structure in rotating frame $\mathcal{R}_1$, we introduce a rotating natural configuration of the structure denoted as Ω_0 in $\mathcal{R}_1$ with boundary $\Gamma_0 \cup \Sigma_0$ (no pre-stresses in the natural configuration). The observation of Ω_0 in $\mathcal{R}_0$ is the domain denoted as $\widetilde{\Omega}_0(t)$ with boundary $\widetilde{\Gamma}_0(t) \cup \widetilde{\Sigma}_0(t)$. The structure is submitted to an external body force field $\widetilde{\rho}(t, \widetilde{\mathbf{x}})\, \widetilde{\mathbf{f}}(t, \widetilde{\mathbf{x}})$ in $\widetilde{\Omega}(t)$ in which $\widetilde{\rho}$ is the mass density of $\widetilde{\Omega}(t)$ and a surface force field $\widetilde{\mathbf{F}}(t, \widetilde{\mathbf{x}})$ on $\widetilde{\Sigma}(t)$. We denote the Cauchy stress tensor related to $\widetilde{\Omega}(t)$ as $\widetilde{\mathbb{O}}$. The observation of domain $\widetilde{\Omega}(t)$ in rotating frame $\mathcal{R}_1$ is the domain denoted as $\Omega(t)$ with boundary $\Gamma(t) \cup \Sigma(t)$. Finally, we have

$$\Omega(t) = \left\{ \mathbf{x} \mid \mathbf{x} = [Q(t)]^T \widetilde{\mathbf{x}} \quad , \quad \widetilde{\mathbf{x}} \in \widetilde{\Omega}(t) \right\} \quad , \tag{2}$$

$$\Sigma(t) = \left\{ \mathbf{x} \mid \mathbf{x} = [Q(t)]^T \widetilde{\mathbf{x}}\, ,\ \widetilde{\mathbf{x}} \in \widetilde{\Sigma}(t) \right\} \quad , \tag{3}$$

$$\Gamma(t) = \left\{ \mathbf{x} \mid \mathbf{x} = [Q(t)]^T \widetilde{\mathbf{x}}\, ,\ \widetilde{\mathbf{x}} \in \widetilde{\Gamma}(t) \right\} \quad , \tag{4}$$

$$\Omega_0 = \left\{ \mathbf{x}_0 \mid \mathbf{x}_0 = [Q(t)]^T \widetilde{\mathbf{x}}_0\, ,\ \widetilde{\mathbf{x}}_0 \in \widetilde{\Omega}_0(t) \right\} \quad , \tag{5}$$

$$\Sigma_0 = \left\{ \mathbf{x}_0 \mid \mathbf{x}_0 = [Q(t)]^T \widetilde{\mathbf{x}}_0\, ,\ \widetilde{\mathbf{x}}_0 \in \widetilde{\Sigma}_0(t) \right\} \quad , \tag{6}$$

$$\Gamma_0 = \left\{ \mathbf{x}_0 \mid \mathbf{x}_0 = [Q(t)]^T \widetilde{\mathbf{x}}_0\, ,\ \widetilde{\mathbf{x}}_0 \in \widetilde{\Gamma}_0(t) \right\} \quad . \tag{7}$$

The displacement field in $\mathcal{R}_1$ of $\Omega(t)$ with respect to Ω_0 is denoted as $\mathbf{u}_0$ and is such that

$$\mathbf{x}(t, \mathbf{x}_0) = \mathbf{x}_0 + \mathbf{u}_0(t, \mathbf{x}_0) \quad , \quad \forall\, \mathbf{x}_0 \in \Omega_0 \quad . \tag{8}$$

Using the methodology presented by Truesdell (1974), we can deduce the equations of the rotating structure for the deformed configuration expressed in $\mathcal{R}_1$, which are written as

$$\forall\, \mathbf{x} \in \Omega(t) \ ,\ \mathbf{div}_{\mathbf{x}} \mathbb{O} + \rho\, \mathbf{f} = \rho \left([R(t)]^2 + [\dot{R}(t)] \right) \mathbf{x} + 2\rho\, [R(t)]\, \dot{\mathbf{x}} + \rho\, \ddot{\mathbf{x}} \quad , \tag{9}$$

$$\forall\, \mathbf{x} \in \Sigma(t) \ ,\ \mathbb{O}\, \mathbf{n} = \mathbf{F} \quad , \tag{10}$$

$$\forall\, \mathbf{x}_0 \in \Gamma_0 \ ,\ \mathbf{u}_0(t, \mathbf{x}_0) = 0 \quad , \tag{11}$$

in which ρ is the mass density of configuration $\Omega(t)$, a dot means the partial derivative with respect to t and where

$$\mathbb{O}(t, \mathbf{x}) = [Q(t)]^T\, \widetilde{\mathbb{O}}(t, \widetilde{\mathbf{x}})\, [Q(t)] \ ,\ \rho(t, \mathbf{x}) = \widetilde{\rho}(t, \widetilde{\mathbf{x}}) \quad ,\ \mathbf{f}(t, \mathbf{x}) = [Q(t)]^T\, \widetilde{\mathbf{f}}(t, \widetilde{\mathbf{x}}) \quad ,$$
$$[R(t)] = [Q(t)]^T [\dot{Q}(t)] \quad ,\ \mathbf{n} = [Q(t)]^T\, \widetilde{\mathbf{n}} \quad ,\ \mathbf{F}(t, \mathbf{x}) = [Q(t)]^T\, \widetilde{\mathbf{F}}(t, \widetilde{\mathbf{x}}) \quad ,$$

where exponent T means the transpose of a matrix. It should be noted that $[R(t)]^T = -[R(t)]$. If the rotation axis and the rotation speed are constants with respect to time t, then $[\dot{R}(t)] = 0$ and $[R(t)]$ is a constant matrix denoted as $[R]$. With this assumption, Eq. (9) can be rewritten as

$$\forall\, \mathbf{x} \in \Omega(t) \quad , \quad \mathbf{div}_{\mathbf{x}} \mathbb{O} + \rho\, \mathbf{f} = \rho\, [R]^2 \mathbf{x} + 2\rho\, [R]\, \dot{\mathbf{x}} + \rho\, \ddot{\mathbf{x}} \quad . \tag{12}$$

The equations can be expressed in terms of unknown $\mathbf{u}_0$ using the Piola identity (Ciarlet, 1988) which is written as $\mathbf{div}_{\mathbf{x}_0} (\mathbb{F}_{\mathbf{u}_0} \mathbb{\Pi}_0) = (\det \mathbb{F}_{\mathbf{u}_0}) \mathbf{div}_{\mathbf{x}} \mathbb{O}$ and Eqs. (10) to (12) yield

$$\forall\, \mathbf{x}_0 \in \Omega_0 \ ,\ \mathbf{div}_{\mathbf{x}_0} (\mathbb{F}_{\mathbf{u}_0} \mathbb{\Pi}_0) + \rho_0\, \mathbf{f}_0 = \rho_0\, [R]^2 \mathbf{x}_0 + \rho_0\, [R]^2 \mathbf{u}_0 + 2\rho_0\, [R]\, \dot{\mathbf{u}}_0 + \rho_0\, \ddot{\mathbf{u}}_0 \quad , \tag{13}$$

$$\forall\, \mathbf{x}_0 \in \Sigma_0 \ ,\ \mathbb{F}_{\mathbf{u}_0} \mathbb{\Pi}_0\, \mathbf{n}_0 = \mathbf{F}_0 \quad , \tag{14}$$

$$\forall\, \mathbf{x}_0 \in \Gamma_0 \ ,\ \mathbf{u}_0(t, \mathbf{x}_0) = 0 \quad , \tag{15}$$

in which

$$\mathbf{f}_0(t,\mathbf{x}_0) = \mathbf{f}(t,\mathbf{x}) \quad , \quad \mathbf{F}_0(t,\mathbf{x}_0)\, dS_0(\mathbf{x}_0) = \mathbf{F}(t,\mathbf{x})\, dS(\mathbf{x}) \quad ,$$
$$\rho_0(\mathbf{x}_0) = \rho(t,\mathbf{x}) \, \det \mathbb{F}_{\mathbf{u}_0} \quad , \quad \mathbb{\Pi}_0 = (\det \mathbb{F}_{\mathbf{u}_0})\, \mathbb{F}_{\mathbf{u}_0}^{-1}\, \mathbb{\sigma}\, \mathbb{F}_{\mathbf{u}_0}^{-T} \quad ,$$
$$\mathbb{F}_{\mathbf{u}_0} = \mathbb{I} + \frac{\partial \mathbf{u}_0}{\partial \mathbf{x}_0} \quad , \quad \mathbb{E}_{\mathbf{u}_0} = \frac{1}{2}\left(\mathbb{F}_{\mathbf{u}_0}^T\, \mathbb{F}_{\mathbf{u}_0} - \mathbb{I}\right) \quad .$$

It should be noted that ρ_0 is the mass density of the rotating natural configuration Ω_0. In this paper, we consider viscoelatic materials and we refer the reader to the theory of linear viscoelasticity in finite displacements developped by B.D. Coleman and V. Noll (1961) using a Lagrangian description. Consequently, the relationship between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor of the deformed configuration expressed with respect to Ω_0 is written as

$$\mathbb{\Pi}_0(t,\mathbf{x}_0) = \mathbb{G}_0(0,\mathbf{x}_0) : \mathbb{E}_{\mathbf{u}_0}(t,\mathbf{x}_0) + \int_0^{+\infty} \dot{\mathbb{G}}_0(s,\mathbf{x}_0) : \mathbb{E}_{\mathbf{u}_0}(t-s,\mathbf{x}_0)\, ds \quad , \tag{16}$$

in which $\{\dot{\mathbb{G}}_0 : \mathbb{E}_{\mathbf{u}_0}\}_{ij} = \{\dot{\mathbb{G}}_0\}_{ijkh}\, \{\mathbb{E}_{\mathbf{u}_0}\}_{kh}$ with summation over k and h, where $t \mapsto \mathbb{G}_0(t,\mathbf{x}_0)$ is the relaxation function defined on $\mathbb{R}$ with support $\mathbb{R}^+$ and values in the fourth-order tensors and where $\dot{\mathbb{G}}_0(t,\mathbf{x}_0)$ denotes the derivative of $\mathbb{G}_0(t,\mathbf{x}_0)$ with respect to t on $]0+\infty[$. If the strucure is only subjected to the stationary parts $\rho_0(\mathbf{x}_0)\mathbf{f}_0^s(\mathbf{x}_0)$ and $\mathbf{F}_0^s(\mathbf{x}_0)$ of the external forces, then the structure is in equilibrium in a stationary configuration represented by the domain denoted as Ω_S in $\mathcal{R}_1$ with boundary $\Gamma_S \cup \Sigma_S$. Let $\mathbf{u}_{0S}(\mathbf{x}_0)$ be the displacement field in $\mathcal{R}_1$ of Ω_S with respect to Ω_0. Let $\mathbb{\Pi}_{0S}$ and $\mathbb{E}_{\mathbf{u}_{0S}}$ be the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor of stationary configuration Ω_S with respect to Ω_0. We have

$$\mathbb{\Pi}_{0S}(\mathbf{x}_0) = \mathbb{G}_0(+\infty,\mathbf{x}_0) : \mathbb{E}_{\mathbf{u}_{0S}}(\mathbf{x}_0) \quad , \quad \forall\, \mathbf{x}_0 \in \Omega_0 \quad . \tag{17}$$

From Eqs. (13) to (15) and Eq. (17), we deduce the variational formulation of the boundary value problem in $\mathbf{u}_{0S}$ for the stationary configuration: find $\mathbf{u}_{0S}$ in the admissible displacement fields $V_0 = \{\mathbf{u} \in \left(H^1(\Omega_0)\right)^3,\ \mathbf{u} = 0 \text{ on } \Gamma_0\}$, such that for all $\delta\mathbf{u}$ in V_0,

$$\int_{\Omega_0} \operatorname{tr}\left\{ (\mathbb{G}_0(+\infty,\mathbf{x}_0) : \mathbb{E}_{\mathbf{u}_{0S}})\, \mathbb{F}_{\mathbf{u}_{0S}}^T \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_0}} \right\} d\mathbf{x}_0 + \int_{\Omega_0} \{\rho_0\, [R]^2\, \mathbf{u}_{0S}\} \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_0$$
$$= \int_{\Omega_0} \rho_0\, \mathbf{f}_0^s(\mathbf{x}_0) \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_0 + \int_{\Sigma_0} \mathbf{F}_0^s(\mathbf{x}_0) \cdot \overline{\delta\mathbf{u}}\, dS_0 - \int_{\Omega_0} \rho_0\, \{[R]^2\, \mathbf{x}_0\} \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_0 \quad . \tag{18}$$

where an overline denotes the conjugate of a complex quantity and where “tr” is the trace operator. Now, the stationary configuration is chosen as the reference configuration. We then introduce in $\mathcal{R}_1$ the Lagrangian transports form Ω_0 into Ω_S and from Ω_S into Ω_0 such that

$$\mathbf{x}_S(\mathbf{x}_0) = \mathbf{x}_0 + \mathbf{u}_{0S}(\mathbf{x}_0) \quad , \quad \forall \mathbf{x}_0 \in \Omega_0 \quad , \tag{19}$$
$$\mathbf{x}_0(\mathbf{x}_S) = \mathbf{x}_S + \mathbf{u}_{S0}(\mathbf{x}_S) \quad , \quad \forall \mathbf{x}_S \in \Omega_S \quad . \tag{20}$$

Let $\mathbb{\Pi}_S$ and $\mathbb{E}_{\mathbf{u}_S}$ be the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor of the deformed configuration $\Omega(t)$ with respect to Ω_S. Consequently, the relationship between $\mathbb{\Pi}_S$ and $\mathbb{E}_{\mathbf{u}_S}$ is written as

$$\mathbb{\Pi}_S(t,\mathbf{x}_S) = \mathbb{\sigma}_S + \mathbb{G}_S(0,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}_S}(t,\mathbf{x}_S) + \int_0^{+\infty} \dot{\mathbb{G}}_S(s,\mathbf{u}_S) : \mathbb{E}_{\mathbf{u}_S}(t-s,\mathbf{x}_S)\, ds \quad , \tag{21}$$

in which

$$\mathbb{O}_S(\mathbf{x}_S) = \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \mathbb{F}_{\mathbf{u}_{0S}} \left(\mathbb{G}_0(+\infty,\mathbf{x}_0) : \mathbb{E}_{\mathbf{u}_{0S}}\right) \mathbb{F}^T_{\mathbf{u}_{0S}} \quad , \tag{22}$$

$$\{\mathbb{G}_S(t,\mathbf{x}_S)\}_{abpq} = \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ak}\{\mathbb{F}_{\mathbf{u}_{0S}}\}_{bl}\{\mathbb{F}_{\mathbf{u}_{0S}}\}_{pm}\{\mathbb{F}_{\mathbf{u}_{0S}}\}_{qn}\{\mathbb{G}_0(t,\mathbf{x}_0)\}_{klmn} \, , \tag{23}$$

and where $\dot{\mathbb{G}}_S(t,\mathbf{x}_0)$ denotes the partial derivative of $\mathbb{G}_S(t,\mathbf{x}_S)$ with respect to t on $]0+\infty[$ (and not on $\mathbb{R}$). Therefore, the viscoelastodynamic equations of the rotating structure with finite displacements is rewritten with respect to this reference configuration using the Piola identity $\mathbf{div}_{\mathbf{x}_S}(\mathbb{F}_{\mathbf{u}_S}\, \mathbb{\Pi}_S) = (\det \mathbb{F}_{\mathbf{u}_S})\mathbf{div}_{\mathbf{x}}\mathbb{O}$. We then deduce the variational formulation of the rotating viscoelastic structure with finite displacements with respect to reference configuration Ω_S: find $\mathbf{u}_S(t,\cdot)$ in the admissible displacement fields $V_S = \{\mathbf{u} \in \left(H^1(\Omega_S)\right)^3 ,\ \mathbf{u} = 0 \text{ on } \Gamma_S\}$, such that for all $\delta\mathbf{u}$ in V_S,

$$\begin{aligned}
&\int_{\Omega_S} \mathrm{tr}\left\{ \mathbb{O}_S \frac{\partial \mathbf{u}_S}{\partial \mathbf{x}_S}^T \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S + \int_{\Omega_S} \mathrm{tr}\left\{ \left(\mathbb{G}_S(0,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}_S}\right) \mathbb{F}^T_{\mathbf{u}_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\
&\quad + \int_{\Omega_S}\int_0^{+\infty} \mathrm{tr}\left\{ \left(\dot{\mathbb{G}}_S(s,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}_S}(t-s,\mathbf{x}_S)\right) \mathbb{F}^T_{\mathbf{u}_S}(t,\mathbf{x}_S) \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} ds\ d\mathbf{x}_S \\
&+ \int_{\Omega_S} \rho_S \left\{[R]^2\, \mathbf{u}_S\right\} \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S + 2\int_{\Omega_S} \rho_S \left\{[R]\, \dot{\mathbf{u}}_S\right\} \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S + \int_{\Omega_S} \rho_S\, \ddot{\mathbf{u}}_S \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S \\
&\qquad = \int_{\Omega_S} \rho_S\, \mathbf{f}^e_S(t,\mathbf{x}_S) \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S + \int_{\Sigma_S} \mathbf{F}^e_S(t,\mathbf{x}_S) \cdot \overline{\delta\mathbf{u}}\, dS_S \quad ,
\end{aligned} \tag{24}$$

in which $\rho_S\, \mathbf{f}^e_S$ and $\mathbf{F}^e_S$ are the time-fluctuation parts around the stationary parts $\rho_S\, \mathbf{f}^s_S$ and $\mathbf{F}^s_S$ defined by $\mathbf{f}_S = \mathbf{f}^e_S + \mathbf{f}^s_S$ and $\mathbf{F}_S = \mathbf{F}^e_S + \mathbf{F}^s_S$ and where $\mathbf{f}_S$ and $\mathbf{F}_S$ are such that $\mathbf{f}_S(t,\mathbf{x}_S) = \mathbf{f}(t,\mathbf{x})$ and $\mathbf{F}_S(t,\mathbf{x}_S)\, dS_S(\mathbf{x}_S) = \mathbf{F}(t,\mathbf{x})\, dS(\mathbf{x})$.

2. ROTATING STRUCTURE PROBLEM LINEARIZED AROUND THE STATIONARY CONFIGURATION.

If we consider small values of displacement field $\mathbf{u}_S(t,\cdot)$, then we can linearize Eq. (24) around the stationary configuration. Using the same notation $\mathbf{u}_S$ for the displacement field of the linearized problem, field $\mathbf{u}_S(t,\cdot)$ belongs to V_S and is such that for all $\delta\mathbf{u}$ in V_S

$$\begin{aligned}
&\int_{\Omega_S} \mathrm{tr}\left\{ \mathbb{O}_S \frac{\partial \mathbf{u}_S}{\partial \mathbf{x}_S}^T \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S + \int_{\Omega_S} \mathrm{tr}\left\{ \left(\mathbb{G}_S(0,\mathbf{x}_S) : \frac{\partial \mathbf{u}_S}{\partial \mathbf{x}_S}\right) \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\
&\quad + \int_{\Omega_S}\int_0^{+\infty} \mathrm{tr}\left\{ \left(\dot{\mathbb{G}}_S(s,\mathbf{x}_S) : \frac{\partial \mathbf{u}_S}{\partial \mathbf{x}_S}(t-s)\right) \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} ds\ d\mathbf{x}_S \\
&+ \int_{\Omega_S} \rho_S \left\{[R]^2\, \mathbf{u}_S\right\} \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S + 2\int_{\Omega_S} \rho_S \left\{[R]\, \dot{\mathbf{u}}_S\right\} \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S + \int_{\Omega_S} \rho_S\, \ddot{\mathbf{u}}_S \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S \\
&\qquad = \int_{\Omega_S} \rho_S\, \mathbf{f}^e_S(t,\mathbf{x}_S) \cdot \overline{\delta\mathbf{u}}\, d\mathbf{x}_S + \int_{\Sigma_S} \mathbf{F}^e_S(t,\mathbf{x}_S) \cdot \overline{\delta\mathbf{u}}\, dS_S \quad .
\end{aligned} \tag{25}$$

Let $\widehat{\mathbf{u}}_S$, $\widehat{\mathbf{f}}^e_S$, $\widehat{\mathbf{F}}^e_S$ be the Fourier transform with respect to t of $\mathbf{u}_S$, $\mathbf{f}^e_S$, $\mathbf{F}^e_S$ and $\widehat{\mathbb{g}}_S$ such that

$$\widehat{\mathbf{u}}_S(\omega,\mathbf{x}_S) = \int_{-\infty}^{+\infty} e^{-i\,\omega\,t}\, \mathbf{u}_S(t,\mathbf{x}_S)\, dt \quad , \quad \widehat{\mathbf{f}}^e_S(\omega,\mathbf{x}_S) = \int_{-\infty}^{+\infty} e^{-i\,\omega\,t}\, \mathbf{f}^e_S(t,\mathbf{x}_S)\, dt \quad ,$$

$$\widehat{\mathbf{F}}^e_S(\omega,\mathbf{x}_S) = \int_{-\infty}^{+\infty} e^{-i\,\omega\,t}\,\mathbf{F}^e_S(t,\mathbf{x}_S)\,dt \quad , \quad \widehat{\mathbb{g}}_S(\omega,\mathbf{x}_S) = \int_0^{+\infty} e^{-i\omega t}\,\dot{\mathbb{G}}_S(t,\mathbf{x}_S)\,dt \quad .$$

The Fourier transforms of external forces $\rho_S\,\mathbf{f}^e_S$ and $\mathbf{F}^e_S$ with respect to t are assumed to be defined as functions (this assumption is coherent due to the centring of external forces around their stationary parts). Let $\mathbb{A}_S$ and $\mathbb{B}_S$ be the elastic and damping tensors with respect to Ω_S, defined by (R. Ohayon & C.Soize, 1998)

$$\mathbb{A}_S(\omega,\mathbf{x}_S) = \mathbb{G}_S(0,\mathbf{x}_S) + \Re e\{\widehat{\mathbb{g}}_S(\omega,\mathbf{x}_S)\} \quad , \quad \omega\,\mathbb{B}_S(\omega,\mathbf{x}_S) = \Im m\{\widehat{\mathbb{g}}_S(\omega,\mathbf{x}_S)\} \quad . \tag{26}$$

Let V'_S be the antidual space of V_S, and $< \cdot,\cdot >$ be the antiduality product between V'_S and V_S. Let $\mathcal{L}(V_S,V'_S)$ be the set of all the bounded linear operators from V_S into V'_S. Assuming that $\mho_S$ is a bounded function in Ω_S and that external forces are sufficiently regular, we introduce the operators $\mathbf{K}_e(\omega)$, $\mathbf{K}_g$, $\mathbf{K}_c$, $\mathbf{D}(\omega)$, $\mathbf{M}$ and $\mathbf{C}$ belonging to $\mathcal{L}(V_S,V'_S)$ and the element $\widehat{\mathbf{f}}^e(\omega)$ belonging to V'_S such that, for all $\mathbf{u}$ and $\delta\mathbf{u}$ in V_S,

$$< \mathbf{K}_e(\omega)\,\mathbf{u},\delta\mathbf{u} >= \int_{\Omega_S} \mathrm{tr}\left\{\left(\mathbb{A}_S(\omega,\mathbf{x}_S) : \frac{\partial\mathbf{u}}{\partial\mathbf{x}_S}\right)\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\} d\mathbf{x}_S \quad , \tag{27}$$

$$< \mathbf{K}_g\,\mathbf{u},\delta\mathbf{u} >= \int_{\Omega_S} \mathrm{tr}\left\{\mho_S\,\frac{\partial\mathbf{u}^T}{\partial\mathbf{x}_S}\,\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\} d\mathbf{x}_S \quad , \tag{28}$$

$$< \mathbf{K}_c\,\mathbf{u},\delta\mathbf{u} >= \int_{\Omega_S} \rho_S\,\{[R]^2\,\mathbf{u}\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \quad , \tag{29}$$

$$< \mathbf{M}\,\mathbf{u},\delta\mathbf{u} >= \int_{\Omega_S} \rho_S\,\mathbf{u}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \quad , \tag{30}$$

$$< \mathbf{D}(\omega)\,\mathbf{u},\delta\mathbf{u} >= \int_{\Omega_S} \mathrm{tr}\left\{\left(\mathbb{B}_S(\omega,\mathbf{x}_S) : \frac{\partial\mathbf{u}}{\partial\mathbf{x}_S}\right)\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\} d\mathbf{x}_S \quad , \tag{31}$$

$$< \mathbf{C}\,\mathbf{u},\delta\mathbf{u} >= 2\int_{\Omega_S} \rho_S\,\{[R]\,\mathbf{u}\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \quad , \tag{32}$$

$$< \widehat{\mathbf{f}}^e(\omega),\delta\mathbf{u} >= \int_{\Omega_S} \rho_S\,\widehat{\mathbf{f}}^e_S(\omega,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S + \int_{\Sigma_S} \widehat{\mathbf{F}}^e_S(\omega,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,dS_S \quad . \tag{33}$$

Note that $\mathbf{K}_e(\omega)$, $\mathbf{D}(\omega)$ and $\mathbf{M}$ are hermitian positive-definite operators, $\mathbf{C}$ is an anti-hermitian operator, $\mathbf{K}_c$ is an hermitian negative operator and $\mathbf{K}_g$ is an hermitian operator. In the low-frequency domain, $\mathbf{K}_e(\omega)$ can be considered as an operator independent of frequency ω. Introducing the operator $\mathbf{K} = \mathbf{K}_e + \mathbf{K}_g + \mathbf{K}_c$ belonging to $\mathcal{L}(V_S,V'_S)$, the linear operator equation corresponding to Eq. (25) is written as

$$\left\{\begin{bmatrix}\mathbf{K} & 0\\ 0 & \mathbf{M}\end{bmatrix} + i\omega\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ -\mathbf{M} & 0\end{bmatrix} + i\omega\begin{bmatrix}\mathbf{D}(\omega) & 0\\ 0 & 0\end{bmatrix}\right\}\begin{bmatrix}\widehat{\mathbf{u}}_S(\omega)\\ i\omega\,\widehat{\mathbf{u}}_S(\omega)\end{bmatrix} = \begin{bmatrix}\widehat{\mathbf{f}}^e(\omega)\\ 0\end{bmatrix} \quad . \tag{34}$$

Introducing $\lambda = -i\omega$, the eigenvalue problem associated with the conservative system is defined as follows: find $(\mathbf{u}_1\,,\,\mathbf{u}_2)$ in $V_S\times V_S$ and $\lambda\in\mathbb{C}$ such that

$$\begin{bmatrix}\mathbf{K} & 0\\ 0 & \mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} = \lambda\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ -\mathbf{M} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} \quad . \tag{35}$$

When $\mathbf{K}$ is a positive-definite operator, then it is shown that the spectrum is constituted of a sequence $\{\lambda_\alpha\}_{\alpha\geq 1}$ such that each eigenvalue λ_α has a finite multiplicity and $|\lambda_\alpha|\rightarrow+\infty$ when $\alpha\rightarrow+\infty$; each eigenvalue can be written as $\lambda_\alpha = -i\,\omega_\alpha$ in which ω_α is a non-zero real number. In addition, if ω_α is a solution then $-\omega_\alpha$ is also a solution. Finally, the familly

$\{(\mathbf{u}_{\alpha,1}\ ,\ \mathbf{u}_{\alpha,2})\}_{\alpha\geq 1}$ of the corresponding eigenfunctions forms a complete set in $V_S \times V_S$ and verifies the following orthogonality conditions

$$< \mathbf{K}\,\mathbf{u}_{\alpha,1},\ \mathbf{u}_{\beta,1} > + < \mathbf{M}\,\mathbf{u}_{\alpha,2},\ \mathbf{u}_{\beta,2} >= \delta_{\alpha\beta} \quad , \tag{36}$$

$$< \mathbf{C}\,\mathbf{u}_{\alpha,1},\ \mathbf{u}_{\beta,1} > + < \mathbf{M}\,\mathbf{u}_{\alpha,2},\ \mathbf{u}_{\beta,1} > - < \mathbf{M}\,\mathbf{u}_{\alpha,1},\ \mathbf{u}_{\beta,2} >= \frac{i}{\omega_\alpha}\delta_{\alpha\beta} \quad , \tag{37}$$

in which $\delta_{\alpha\beta}$ is the Kronecker symbol. If $(\widehat{\mathbf{u}}_S\ ,\ i\omega\,\widehat{\mathbf{u}}_S)$ is expanded as

$$\begin{bmatrix} \widehat{\mathbf{u}}_S \\ i\omega\,\widehat{\mathbf{u}}_S \end{bmatrix} = \sum_{\alpha=1}^{+\infty} U_\alpha \begin{bmatrix} \mathbf{u}_{\alpha,1} \\ \mathbf{u}_{\alpha,2} \end{bmatrix} \quad , \tag{38}$$

then, for all integer $\beta \geq 1$, we have

$$\left(1 - \frac{\omega}{\omega_\beta}\right) U_\beta + i\omega \sum_{\alpha=1}^{+\infty} < \mathbf{D}(\omega)\,\mathbf{u}_{\alpha,1}, \mathbf{u}_{\beta,1} > U_\alpha \ = < \widehat{\mathbf{f}}^e(\omega), \mathbf{u}_{\beta,1} > \quad . \tag{39}$$

3. PERTURBED NONLINEAR EQUATION IN THE ROTATING FRAME WITH RESPECT TO THE STATIONARY CONFIGURATION.

3.1 Perturbations of the rotating natural configuration Ω_0.

In this section, we define the perturbations of the natural configuration Ω_0 observed in the rotating frame. In a first step, we introduce a domain $\Omega_{\overline{0}}$ as a domain occupied by the perturbed natural configuration observed in $\mathcal{R}_1$. We define the geometrical perturbations as a displacement field in $\mathcal{R}_1$ denoted as $\mathbf{u}_{0\overline{0}}$ and defined on Ω_0. We have

$$\Omega_{\overline{0}} = \{\mathbf{x}_{\overline{0}} \quad , \quad \mathbf{x}_{\overline{0}} = \mathbf{x}_0 + \mathbf{u}_{0\overline{0}}(\mathbf{x}_0) \quad , \quad \forall\, \mathbf{x}_0 \in \Omega_0\} \quad . \tag{40}$$

In a second step, we transport mass density ρ_0 and relaxation function $\mathbb{G}_0$ from Ω_0 to $\Omega_{\overline{0}}$ which are denoted as $\rho_{\overline{0}}$ and $\mathbb{G}_{\overline{0}}$ respectively. We have

$$\rho_{\overline{0}}(\mathbf{x}_0 + \mathbf{u}_{0\overline{0}}(\mathbf{x}_0)) = \rho_0(\mathbf{x}_0) \quad , \quad \mathbb{G}_{\overline{0}}(t, \mathbf{x}_0 + \mathbf{u}_{0\overline{0}}(\mathbf{x}_0)) = \mathbb{G}_0(t, \mathbf{x}_0) \quad . \tag{41}$$

In a last step, we introduce the perturbations $\Delta\rho_{\overline{0}}$ and $\Delta\mathbb{G}_{\overline{0}}$ of $\rho_{\overline{0}}$ and $\mathbb{G}_{\overline{0}}$ with respect to $\Omega_{\overline{0}}$

$$\rho'_{\overline{0}} = \rho_{\overline{0}} + \Delta\rho_{\overline{0}} \quad , \quad \mathbb{G}'_{\overline{0}} = \mathbb{G}_{\overline{0}} + \Delta\mathbb{G}_{\overline{0}} \quad . \tag{42}$$

In practice, the perturbations $\Delta\rho_{\overline{0}}$ and $\Delta\mathbb{G}_{\overline{0}}$ of the mass density and the relaxation function have to be defined with respect to domain Ω_0. Consequently, we proceed to a Lagrangian transport of perturbations $\Delta\rho_{\overline{0}}$ and $\Delta\mathbb{G}_{\overline{0}}$ from $\Omega_{\overline{0}}$ to Ω_0 and we use the same notation. Due to this perturbations, stationary configuration Ω_S is modified and occupies a domain denoted as $\Omega_{\overline{S}}$. Nevertheless, we do not consider the viscoelastodynamic equations of the perturbed configuration around $\Omega_{\overline{S}}$ but we consider it around Ω_S because, in practice, all the calculations are performed with respect to the stationary configuration of the unperturbed structure. In $\mathcal{R}_1$ and at time t, the deformed configuration of the perturbed structure occupies a domain denoted as $\Omega'(t)$. We then introduce new displacement field $\mathbf{u}'_S(t,\cdot)$ which represents the displacement field of $\Omega'(t)$ with respect to Ω_S,

$$\mathbf{x}'(t, \mathbf{x}_S) = \mathbf{x}_S + \mathbf{u}'_S(t, \mathbf{x}_S) \quad , \quad \forall \mathbf{x}_S \in \Omega_S \quad . \tag{43}$$

3.2 Weakly perturbed nonlinear equations in the rotating frame with respect to the unperturbed stationary configuration.

If the perturbations are sufficiently small, then the nonlinear equations of the perturbed structure in the rotating frame (with respect to the the unperturbed stationary configuration) can be linearized with respect to perturbations $\mathbf{u}_{0\overline{0}}$, $\Delta\rho_{\overline{0}}$ and $\Delta\mathbb{G}_{\overline{0}}$ (we do not consider the linearization with respect to displacement field $\mathbf{u}'_S$). In the following equations, each quantity which is related to Ω_S and which is expressed as a function of $\mathbf{x}_0$ has to be submitted to the Lagrangian transport defined by Eq. (20). In order to simplify the equations we introduce the tensors $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{K}_1$ and $\mathbb{K}_2$ such that

$$\{\mathbb{K}_1(\mathbf{x}_S)\}_{ijpqab} = \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ia} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{jb} \{\mathbb{E}_{\mathbf{u}_{0S}}\}_{pq} \quad , \tag{44}$$

$$\begin{aligned}\{\mathbb{T}_1(\mathbf{x}_S)\}_{ijkh} = &-\frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ia} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{jh} \{\mathbb{E}_{\mathbf{u}_{0S}}\}_{pq} \{\mathbb{G}_0(+\infty,\mathbf{x}_0)\}_{akpq} \\ &- \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ja} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ih} \{\mathbb{E}_{\mathbf{u}_{0S}}\}_{pq} \{\mathbb{G}_0(+\infty,\mathbf{x}_0)\}_{akpq} \\ &- \frac{2}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ia} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{jb} \{\mathbb{E}_{\mathbf{u}_{0S}}\}_{ph} \{\mathbb{G}_0(+\infty,\mathbf{x}_0)\}_{abpk} \\ &- \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ia} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{jb} \{\mathbb{G}_0(+\infty,\mathbf{x}_0)\}_{akpq} \quad ,\end{aligned} \tag{45}$$

$$\{\mathbb{K}_2(\mathbf{x}_S)\}_{abpqklmn} = \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ak} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{bl} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{pm} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{qn} \quad , \tag{46}$$

$$\begin{aligned}\{\mathbb{T}_2(t,\mathbf{x}_S)\}_{abpqij} = &-\frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ak} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{bj} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{pm} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{qn} \{\mathbb{G}_0(t,\mathbf{x}_0)\}_{kimn} \\ &- \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{aj} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{bl} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{pm} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{qn} \{\mathbb{G}_0(t,\mathbf{x}_0)\}_{ilmn} \\ &- \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ak} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{bl} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{pj} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{qn} \{\mathbb{G}_0(t,\mathbf{x}_0)\}_{klin} \\ &- \frac{1}{\det \mathbb{F}_{\mathbf{u}_{0S}}} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{ak} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{bl} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{pm} \{\mathbb{F}_{\mathbf{u}_{0S}}\}_{qj} \{\mathbb{G}_0(t,\mathbf{x}_0)\}_{klmi} \quad .\end{aligned} \tag{47}$$

Therefore, displacement field $\mathbf{u}'_S(t,\cdot)$ in V_S is such that for all $\delta\mathbf{u}$ in V_S,

$$\begin{aligned}&\int_{\Omega_S} \mathrm{tr}\left\{ \mathbb{O}_S \frac{\partial \mathbf{u}'^T_S}{\partial \mathbf{x}_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S + \int_{\Omega_S} \mathrm{tr}\left\{ \left(\mathbb{K}_1 :: \Delta\mathbb{G}_{\overline{0}}(+\infty,\mathbf{x}_0)\right) \frac{\partial \mathbf{u}'^T_S}{\partial \mathbf{x}_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\ &+ \int_{\Omega_S} \mathrm{tr}\left\{ \left(\mathbb{T}_1 : \frac{\partial \mathbf{u}_{0\overline{0}}}{\partial \mathbf{x}_0}\right) \frac{\partial \mathbf{u}'^T_S}{\partial \mathbf{x}_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S + \int_{\Omega_S} (\mathrm{div}_{\mathbf{x}_0} \mathbf{u}_{0\overline{0}})\, \mathrm{tr}\left\{ \mathbb{O}_S \frac{\partial \mathbf{u}'^T_S}{\partial \mathbf{x}_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\ &+ \int_{\Omega_S} \mathrm{tr}\left\{ \left(\mathbb{G}_S(0,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}'_S}\right) \mathbb{F}^T_{\mathbf{u}'_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\ &+ \int_{\Omega_S} (\mathrm{div}_{\mathbf{x}_0} \mathbf{u}_{0\overline{0}})\, \mathrm{tr}\left\{ \left(\mathbb{G}_S(0,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}'_S}\right) \mathbb{F}^T_{\mathbf{u}'_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\ &+ \int_{\Omega_S} \mathrm{tr}\left\{ \left[\left(\mathbb{T}_2(0,\mathbf{x}_S) : \frac{\partial \mathbf{u}_{0\overline{0}}}{\partial \mathbf{x}_0}\right) : \mathbb{E}_{\mathbf{u}'_S}\right] \mathbb{F}^T_{\mathbf{u}'_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\ &+ \int_{\Omega_S} \mathrm{tr}\left\{ \left[\left(\mathbb{K}_2 :: \Delta\mathbb{G}_{\overline{0}}(0,\mathbf{x}_0)\right) : \mathbb{E}_{\mathbf{u}'_S}\right] \mathbb{F}^T_{\mathbf{u}'_S} \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} d\mathbf{x}_S \\ &+ \int_{\Omega_S} \int_0^{+\infty} \mathrm{tr}\left\{ \left(\dot{\mathbb{G}}_S(s,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}'_S}(t-s,\mathbf{x}_S)\right) \mathbb{F}^T_{\mathbf{u}'_S}(t,\mathbf{x}_S) \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} ds\, d\mathbf{x}_S \\ &+ \int_{\Omega_S} \int_0^{+\infty} (\mathrm{div}_{\mathbf{x}_0} \mathbf{u}_{0\overline{0}})\, \mathrm{tr}\left\{ \left(\dot{\mathbb{G}}_S(s,\mathbf{x}_S) : \mathbb{E}_{\mathbf{u}'_S}(t-s,\mathbf{x}_S)\right) \mathbb{F}^T_{\mathbf{u}'_S}(t,\mathbf{x}_S) \overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}} \right\} ds\, d\mathbf{x}_S\end{aligned}$$

$$+\int_{\Omega_S}\int_0^{+\infty}\text{tr}\left\{\left[\left(\dot{\mathbb{T}}_2(s,\mathbf{x}_S):\frac{\partial\mathbf{u}_{0\overline{0}}}{\partial\mathbf{x}_0}\right):\mathbb{E}_{\mathbf{u}'_S}(t-s,\mathbf{x}_S)\right]\mathbb{F}^T_{\mathbf{u}'_S}(t,\mathbf{x}_S)\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\}ds\;d\mathbf{x}_S$$

$$+\int_{\Omega_S}\int_0^{+\infty}\text{tr}\left\{\left[\left(\mathbb{K}_2::\Delta\dot{\mathbb{G}}_{\overline{0}}(s,\mathbf{x}_0)\right):\mathbb{E}_{\mathbf{u}'_S}(t-s,\mathbf{x}_S)\right]\mathbb{F}^T_{\mathbf{u}'_S}(t,\mathbf{x}_S)\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\}ds\;d\mathbf{x}_S$$

$$+\int_{\Omega_S}\rho_S\ddot{\mathbf{u}}'_S\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S+\int_{\Omega_S}\rho_S\,(\text{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}})\;\ddot{\mathbf{u}}'_S\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_{0S}}}\ddot{\mathbf{u}}'_S\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S+\int_{\Omega_S}\rho_S\{[R]^2\mathbf{u}'_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+\int_{\Omega_S}\rho_S\,(\text{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}})\;\{[R]^2\mathbf{u}'_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S+\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_{0S}}}\{[R]^2\mathbf{u}'_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+2\int_{\Omega_S}\rho_S\{[R]\dot{\mathbf{u}}'_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S+2\int_{\Omega_S}\rho_S\,(\text{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}})\;\{[R]\dot{\mathbf{u}}'_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+2\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_{0S}}}\{[R]\dot{\mathbf{u}}'_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S=<\mathbf{f}'(t),\delta\mathbf{u}>-<\mathbf{f}^c(t),\delta\mathbf{u}>\quad,\qquad(48)$$

in which

$$\{\mathbb{K}_1(\mathbf{x}_S)::\Delta\mathbb{G}_{\overline{0}}(+\infty,\mathbf{x}_0)\}_{ij}=\{\mathbb{K}_1(\mathbf{x}_S)\}_{ijpqab}\,\{\Delta\mathbb{G}_{\overline{0}}(+\infty,\mathbf{x}_0)\}_{abpq}\quad,\qquad(49)$$

$$\{\mathbb{K}_2(\mathbf{x}_S)::\Delta\dot{\mathbb{G}}_{\overline{0}}(s,\mathbf{x}_0)\}_{ijpq}=\{\mathbb{K}_2(\mathbf{x}_S)\}_{ijpqklmn}\,\{\dot{\mathbb{G}}_{\overline{0}}(s,\mathbf{x}_0)\}_{klmn}\quad.\qquad(50)$$

and where $\mathbf{f}'(t)$ and $\mathbf{f}^c(t)$ are the elements belonging to V'_S such that for all $\delta\mathbf{u}$ in V_S,

$$<\mathbf{f}'(t),\delta\mathbf{u}>=\int_{\Sigma_S}\mathbf{F}'_S(t,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,dS_S+\int_{\Omega_S}\rho_S\;\mathbf{f}'_S(t,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+\int_{\Omega_S}\rho_S\,(\text{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}})\;\mathbf{f}'_S(t,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S+\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_{0S}}}\mathbf{f}'_S(t,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S\quad,\qquad(51)$$

$$<\mathbf{f}^c,\delta\mathbf{u}>=\int_{\Sigma_S}\mathbf{F}^s_S(\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,dS_S+\int_{\Omega_S}\rho_S\;\mathbf{f}^s_S(\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+\int_{\Omega_S}\rho_S\,(\text{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}})\;\{[R]^2\mathbf{x}_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S+\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_{0S}}}\{[R]^2\mathbf{x}_S\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S$$

$$+\int_{\Omega_S}(\text{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}})\;\text{tr}\left\{\mathbb{O}_S\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\}d\mathbf{x}_S+\int_{\Omega_S}\text{tr}\left\{\left(\mathbb{T}_1:\frac{\partial\mathbf{u}_{0\overline{0}}}{\partial\mathbf{x}_0}\right)\;\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\}d\mathbf{x}_S$$

$$+\int_{\Omega_S}\text{tr}\left\{\left(\mathbb{K}_1::\Delta\mathbb{G}_{\overline{0}}(+\infty,\mathbf{x}_0)\right)\;\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\}d\mathbf{x}_S\quad,\qquad(52)$$

In Eq. (51), $\mathbf{f}'_S$ and $\mathbf{F}'_S$ are defined by

$$\mathbf{F}'_S(t,\mathbf{x}_S)\,dS_S(\mathbf{x}_S)=[Q(t)]\,\mathbf{F}(t,\mathbf{x}')\,dS(\mathbf{x}')\quad,\quad\mathbf{f}'_S(t,\mathbf{x}_S)=[Q(t)]\,\mathbf{f}(t,\mathbf{x}')\quad.\qquad(53)$$

in which $\mathbf{x}'(t,\mathbf{x}_S)=\mathbf{x}_S+\mathbf{u}'_S(t,\mathbf{x}_S)$.

3.3 Linearization of the perturbed nonlinear equations in the rotating frame around the unperturbed stationary configuration.

In a first step, we introduce the operators $\mathbf{P}_G(\omega)$, $\mathbf{P}_M(\omega)$, $\mathbf{P}_C(\omega)$ belonging to $\mathcal{L}(V_S,V'_S)$ such that

$$<\mathbf{P}_G(\omega)\,\mathbf{u},\delta\mathbf{u}>=\int_{\Omega_S}\text{tr}\left\{\left(\mathbb{T}_1:\frac{\partial\mathbf{u}_{0\overline{0}}}{\partial\mathbf{x}_0}\right)\frac{\partial\mathbf{u}^T}{\partial\mathbf{x}_S}\overline{\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}_S}}\right\}d\mathbf{x}_S$$

$$
\begin{aligned}
&+\int_{\Omega_S} \left(\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\right)\ \mathrm{tr}\left\{\mathbb{C}_S\,\frac{\partial \mathbf{u}^T}{\partial \mathbf{x}_S}\,\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&+\int_{\Omega_S} \left(\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\right) \mathrm{tr}\left\{\left(\mathbb{A}_S : \frac{\partial \mathbf{u}}{\partial \mathbf{x}_S}\right)\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&+i\,\omega\int_{\Omega_S} \left(\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\right)\ \mathrm{tr}\left\{\left(\mathbb{B}_S(\omega,\mathbf{x}_S) : \frac{\partial \mathbf{u}}{\partial \mathbf{x}_S}\right)\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&+\int_{\Omega_S} \mathrm{tr}\left\{\left[\left(\mathbb{T}_2(0,\mathbf{x}_S) : \frac{\partial \mathbf{u}_{0\overline{0}}}{\partial \mathbf{x}_0}\right) : \frac{\partial \mathbf{u}}{\partial \mathbf{x}_S}\right]\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&+\int_{\Omega_S} \mathrm{tr}\left\{\left[\left(\widehat{\mathbb{t}}_2(\omega,\mathbf{x}_S) : \frac{\partial \mathbf{u}_{0\overline{0}}}{\partial \mathbf{x}_0}\right) : \frac{\partial \mathbf{u}}{\partial \mathbf{x}_S}\right]\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&-\omega^2\int_{\Omega_S} \rho_S\left(\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\right)\ \mathbf{u}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S + \int_{\Omega_S}\rho_S\left(\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\right)\ \{[R]^2\mathbf{u}\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \\
&+2i\,\omega\int_{\Omega_S}\rho_S\{\left(\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\right)\ [R]\,\mathbf{u}\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \quad ,
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
<\mathbf{P}_M(\omega)\,\mathbf{u},\delta\mathbf{u}> &= \int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_0 S}}\{[R]^2\mathbf{u}\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S - \omega^2\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_0 S}}\mathbf{u}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \\
&+2i\,\omega\int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_0 S}}\{[R]\,\mathbf{u}\}\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \quad ,
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
<\mathbf{P}_C(\omega)\,\mathbf{u},\delta\mathbf{u}> &= \int_{\Omega_S}\mathrm{tr}\left\{\left(\mathbb{K}_1 :: \Delta\mathbb{G}_{\overline{0}}(+\infty,\mathbf{x}_0)\right)\frac{\partial \mathbf{u}^T}{\partial \mathbf{x}_S}\,\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&+\int_{\Omega_S}\mathrm{tr}\left\{\left[\left(\mathbb{K}_2 :: \Delta\mathbb{G}_{\overline{0}}(0,\mathbf{x}_0)\right) : \frac{\partial \mathbf{u}}{\partial \mathbf{x}_S}\right]\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \\
&+\int_{\Omega_S}\mathrm{tr}\left\{\left[\left(\mathbb{K}_2 :: \Delta\widehat{\mathbb{g}}_{\overline{0}}(\omega,\mathbf{x}_0)\right) : \frac{\partial \mathbf{u}}{\partial \mathbf{x}_S}\right]\overline{\frac{\partial \delta\mathbf{u}}{\partial \mathbf{x}_S}}\right\} d\mathbf{x}_S \quad ,
\end{aligned}
\tag{56}
$$

in which $\widehat{\mathbb{t}}_2(\omega,\mathbf{x}_S)$ and $\Delta\widehat{\mathbb{g}}_0(\omega,\mathbf{x}_S)$ are such that

$$
\widehat{\mathbb{t}}_2(\omega,\mathbf{x}_S) = \int_0^{+\infty} e^{-i\,\omega\,t}\,\dot{\mathbb{T}}_2(t,\mathbf{x}_S)\,dt \quad , \quad \Delta\widehat{\mathbb{g}}_{\overline{0}}(\omega,\mathbf{x}_0) = \int_0^{+\infty} e^{-i\,\omega\,t}\,\Delta\dot{\mathbb{G}}_{\overline{0}}(t,\mathbf{x}_0)\,dt \quad .
$$

In a second step, in the context of linearization, we write displacement field $\mathbf{u}'_S$ as

$$
\mathbf{u}'_S(t,\mathbf{x}_S) = \mathbf{u}^{s\prime}_S(\mathbf{x}_S) + \mathbf{u}^{e\prime}_S(t,\mathbf{x}_S) \quad , \tag{57}
$$

in which $\mathbf{u}^{s\prime}_S$ in V_S is such that for all $\delta\mathbf{u}$ in V_S,

$$
\begin{aligned}
&<\mathbf{K}\,\mathbf{u}^{s\prime}_S,\delta\mathbf{u}> + <\mathbf{P}_G(0)\,\mathbf{u}^{s\prime}_S,\delta\mathbf{u}> + <\mathbf{P}_M(0)\,\mathbf{u}^{s\prime}_S,\delta\mathbf{u}> + <\mathbf{P}_C(0)\,\mathbf{u}^{s\prime}_S,\delta\mathbf{u}> \\
&\qquad = \int_{\Sigma_S}\mathbf{F}^{s\prime}_S(\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,dS_S + \int_{\Omega_S}\rho_S\,\mathbf{f}^{s\prime}_S(\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S - <\mathbf{f}^c,\delta\mathbf{u}> \\
&\quad + \int_{\Omega_S}\rho_S\mathrm{div}_{\mathbf{x}_0}\mathbf{u}_{0\overline{0}}\,\mathbf{f}^{s\prime}_S(t,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S + \int_{\Omega_S}\frac{\Delta\rho_{\overline{0}}}{\det\mathbb{F}_{\mathbf{u}_0 S}}\mathbf{f}^{s\prime}_S(t,\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S \quad ,
\end{aligned}
\tag{58}
$$

where $\mathbf{f}^{s\prime}_S$ and $\mathbf{F}^{s\prime}_S$ are the stationary parts of fields $\mathbf{f}'_S$ and $\mathbf{F}'_S$. Let $\mathbf{f}^{e\prime}(t)$ be the element belonging to V'_S such that for all $\delta\mathbf{u}$ in V_S

$$
<\mathbf{f}^{e\prime}(t),\delta\mathbf{u}> = <\mathbf{f}'(t),\delta\mathbf{u}> - \int_{\Sigma_S}\mathbf{F}^{s\prime}_S(\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,dS_S - \int_{\Omega_S}\rho_S\,\mathbf{f}^{s\prime}_S(\mathbf{x}_S)\cdot\overline{\delta\mathbf{u}}\,d\mathbf{x}_S
$$

$$- \int_{\Omega_S} \rho_S \mathrm{div}_{\mathbf{x}_0} \mathbf{u}_{0\overline{0}}\, \mathbf{f}_S^{s\prime}(\mathbf{x}_S) \cdot \overline{\delta \mathbf{u}}\, d\mathbf{x}_S - \int_{\Omega_S} \frac{\Delta \rho_{\overline{0}}}{\det \mathbb{F}_{\mathbf{u}_0 S}} \mathbf{f}_S^{s\prime}(\mathbf{x}_S) \cdot \overline{\delta \mathbf{u}}\, d\mathbf{x}_S \quad . \quad (59)$$

It should be noted that field $\mathbf{u}_S^{s\prime}$ corresponds to the displacement field of $\Omega_{\overline{S}}$ with respect to Ω_S calculated with the linearized equation with respect to the displacement field. Let $\widehat{\mathbf{u}}_S^{e\prime}$ and $\widehat{\mathbf{f}}^{e\prime}(\omega)$ be the Fourier transform with respect to t of $\mathbf{u}_S^{e\prime}$ and $\mathbf{f}^{e\prime}(t)$ such that

$$\widehat{\mathbf{u}}_S^{e\prime}(\omega, \mathbf{x}_S) = \int_{-\infty}^{+\infty} e^{-i\,\omega\, t}\, \mathbf{u}_S^{e\prime}(t, \mathbf{x}_S)\, dt\,, < \widehat{\mathbf{f}}^{e\prime}(\omega), \delta \mathbf{u} >= \int_{-\infty}^{+\infty} e^{-i\,\omega\, t} < \mathbf{f}^{e\prime}(t), \delta \mathbf{u} > dt\,.$$

Then for small strains and in the frequency domain, the linear operator equation corresponding to Eq. (48) is written as

$$\left[\mathbf{K} + i\,\omega\,\mathbf{C} + i\,\omega\,\mathbf{D}(\omega) - \omega^2\,\mathbf{M} + \mathbf{P}_G(\omega) + \mathbf{P}_M(\omega) + \mathbf{P}_C(\omega)\right]\, \widehat{\mathbf{u}}_S^{e\prime}(\omega, \mathbf{x}_s) = \mathbf{f}^{e\prime}(\omega)\,. \quad (60)$$

Since $\{(\mathbf{u}_{\alpha,1}\ ,\ \mathbf{u}_{\alpha,2})\}_{\alpha \geq 1}$ forms a complete set in $V_S \times V_S$, then $(\widehat{\mathbf{u}}_S^{e\prime}\ ,\ i\omega\,\widehat{\mathbf{u}}_S^{e\prime})$ can be expanded as

$$\begin{bmatrix} \widehat{\mathbf{u}}_S^{e\prime} \\ i\omega\,\widehat{\mathbf{u}}_S^{e\prime} \end{bmatrix} = \sum_{\alpha=1}^{+\infty} U'_\alpha\, \begin{bmatrix} \mathbf{u}_{\alpha,1} \\ \mathbf{u}_{\alpha,2} \end{bmatrix} \quad . \quad (61)$$

Therefore, Eq. (60) can be rewritten such that for all integer $\beta \geq 1$,

$$\left(1 - \frac{\omega}{\omega_\beta}\right) U'_\beta + i\,\omega\, \sum_{\alpha=1}^{+\infty} < \mathbf{D}(\omega)\, \mathbf{u}_{\alpha,1}, \mathbf{u}_{\beta,1} >\ U'_\alpha\ + \sum_{\alpha=1}^{+\infty} < \mathbf{P}_G(\omega)\, \mathbf{u}_{\alpha,1}, \mathbf{u}_{\beta,1} >\ U'_\alpha$$

$$+ \sum_{\alpha=1}^{+\infty} < \mathbf{P}_M(\omega)\, \mathbf{u}_{\alpha,1}, \mathbf{u}_{\beta,1} >\ U'_\alpha\ + \sum_{\alpha=1}^{+\infty} < \mathbf{P}_C(\omega)\, \mathbf{u}_{\alpha,1}, \mathbf{u}_{\beta,1} >\ U'_\alpha$$

$$= < \widehat{\mathbf{f}}^{e\prime}(\omega), \mathbf{u}_{\beta,1} > \quad . \quad (62)$$

4. CONCLUSION.

We have presented the nonlinear structural dynamic equations in finite displacements of a rotating structure expressed with respect to a pre-stressed configuration Ω_S. We have considered perturbations of the natural configuration of the structure. Then, we have written the nonlinear viscoelastodynamic equations of the weakly perturbed structure in finite displacement. For small strains around the stationary configuration, the equations of the weakly perturbed rotating structure are linearized around stationary configuration Ω_S. Then the solution of the perturbed problem is expanded on the eigenfunctions of the unperturbed rotating structure. We show that the equations obtained are not self-evident, particulary for geometrical perturbations.

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